Name of College - S. B. College, J. Bodh.

Deptt - Mathematics

TEACHER'S NAME - SAMRENDRA KUMAR.

TOPIC - Problem based on Inverse Hyperbolic Functions (Contd)

B.Sc.I (HONS)

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By - Samrendra Kumar.

Contd

Problem :- Show that

$$\tan^{-1}(\cos\theta + i\sin\theta) = \frac{n\pi}{2} + \frac{\pi}{4} + \frac{i}{2}\log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

Solution :- Let $\tan^{-1}(\cos\theta + i\sin\theta) = \alpha + i\beta$

$$\Rightarrow \tan(\alpha + i\beta) = \cos\theta + i\sin\theta$$

$$\Rightarrow \tan(\alpha - i\beta) = \cos\theta - i\sin\theta.$$

Now $2\alpha = [(\alpha + i\beta) + (\alpha - i\beta)]$

$$\tan 2\alpha = \tan[(\alpha + i\beta) + (\alpha - i\beta)]$$

$$= \frac{\tan(\alpha + i\beta) + \tan(\alpha - i\beta)}{1 - \tan(\alpha + i\beta)\cdot\tan(\alpha - i\beta)}$$

$$= \frac{\cos\theta + i\sin\theta + \cos\theta - i\sin\theta}{1 - (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}$$

$$= \frac{2\cos\theta}{1 - [\cos^2\theta - i^2\sin^2\theta]}$$

$$= \frac{2\cos\theta}{1 - (\cos^2\theta + \sin^2\theta)} = \frac{2\cos\theta}{0} = \infty = \tan\pi/2$$

$$\left[\because \tan\theta = \tan\alpha \Rightarrow \theta = n\pi/2 + \alpha\right]$$

$$\therefore 2\alpha = n\pi + \pi/2$$

Also $2i\beta = [(\alpha+i\beta) - (\alpha-i\beta)]$

$\Rightarrow \tan 2i\beta = \tan[(\alpha+i\beta) - (\alpha-i\beta)]$

$$\Rightarrow i\tanh 2\beta = \frac{\tan(\alpha+i\beta) - \tan(\alpha-i\beta)}{1+\tan(\alpha+i\beta)\cdot\tan(\alpha-i\beta)}$$

$$= \frac{\cos\theta + i\sin\theta - \cos\theta + i\sin\theta}{1+\cos^2\theta+\sin^2\theta}$$

$$= \frac{2i\sin\theta}{2}$$

$\Rightarrow i\tanh 2\beta = i\sin\theta$

$\Rightarrow \tanh 2\beta = \sin\theta$

$\Rightarrow 2\beta = \tanh^{-1}(\sin\theta)$

$$= \frac{1}{2}\log\frac{1+\sin\theta}{1-\sin\theta}$$

$$= \frac{1}{2}\log\left[\frac{\cos\theta/2+\sin\theta/2}{\cos\theta/2-\sin\theta/2}\right]^2$$

$$= \frac{1}{2}\times 2.\log\frac{\cos\theta/2+\sin\theta/2}{\cos\theta/2-\sin\theta/2}$$

$$= \log\frac{1+\tan\theta/2}{1-\tan\theta/2}$$

$\therefore 2\beta = \log\tan\left(\frac{\pi}{4}+\theta/2\right)$

$\Rightarrow \beta = \frac{1}{2}\log\tan\left(\frac{\pi}{4}+\theta/2\right)$

Hence $\tan^{-1}(\cos\theta+i\sin\theta) = \frac{n\pi}{2}+\frac{\pi}{4}+\frac{i}{2}\log\left|\tan\left(\frac{\pi}{4}+\theta/2\right)\right|$

$\therefore$ proved

Separate $\cos^{-1}(\alpha+i\beta)$ into real and imaginary parts.

Solution: let $\cos^{-1}(\alpha+i\beta) = x+iy \Rightarrow \cos(x+iy) = \alpha+i\beta$

$\Rightarrow \cos^{-1}(\alpha-i\beta) = x-iy \Rightarrow \cos(x-iy) = \alpha-i\beta$

$$2x = (x+iy) + (x-iy)$$

$$\cos 2x = \cos[(x+iy)+(x-iy)]$$

$$= \cos(x+iy)\cos(x-iy) - \sin(x+iy)\sin(x-iy)$$

$$= (\alpha+i\beta)(\alpha-i\beta) - \sqrt{1-(\alpha+i\beta)^2}\sqrt{1-(\alpha-i\beta)^2}$$

$$= (\alpha^2-i^2\beta^2) - \left[\sqrt{1-\alpha^2+\beta^2-2i\alpha\beta}\right]\sqrt{(1-\alpha^2+\beta^2)+2i\alpha\beta}$$

$$= (\alpha^2+\beta^2) - \sqrt{(1-\alpha^2+\beta^2)^2-4i^2\alpha^2\beta^2}$$

$$= (\alpha^2+\beta^2) - \sqrt{(1-\alpha^2+\beta^2)^2+4\alpha^2\beta^2}$$

$$\therefore 2x = \cos^{-1}\left[\alpha^2+\beta^2 - \sqrt{(1-\alpha^2+\beta^2)^2+4\alpha^2\beta^2}\right]$$

$$\Rightarrow x = \frac{1}{2}\cos^{-1}\left[(\alpha^2+\beta^2) - \sqrt{(1-\alpha^2+\beta^2)^2+4\alpha^2\beta^2}\right]$$

Similarly

$$\cos 2iy = \alpha^2+\beta^2 + \sqrt{(1-\alpha^2+\beta^2)^2+4\alpha^2\beta^2}$$

$$\Rightarrow \cosh 2y = \alpha^2+\beta^2 + \sqrt{(1-\alpha^2+\beta^2)^2+4\alpha^2\beta^2}$$

$$\Rightarrow 2y = \cosh^{-1}\left[\alpha^2+\beta^2 + \sqrt{(1-\alpha^2+\beta^2)^2+4\alpha^2\beta^2}\right]$$

$$\Rightarrow y = \frac{1}{2}\cosh^{-1}\left[\alpha^2+\beta^2 + \sqrt{(1-\alpha^2+\beta^2)^2+4\alpha^2\beta^2}\right]$$

Separate into Real and imaginary parts the Quantity $\tan^{-1}(\alpha+i\beta)$

Solution :-> Let $\tan^{-1}(\alpha+i\beta) = x+iy$.

[Clearly Real part $= x$
Imaginary part $= y$]

$$\Rightarrow \tan(x+iy) = \alpha+i\beta$$

$$\Rightarrow \tan(x-iy) = \alpha-i\beta$$

$$2x = (x+iy) + [x-iy]$$

$$\Rightarrow \tan 2x = \tan[(x+iy)+(x-iy)]$$

$$= \frac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\cdot\tan(x-iy)}$$

$$= \frac{\alpha+i\beta+\alpha-i\beta}{1-(\alpha+i\beta)(\alpha-i\beta)}$$

$$= \frac{2\alpha}{1-\alpha^2-\beta^2}$$

$$\therefore 2x = \tan^{-1}\frac{2\alpha}{1-\alpha^2-\beta^2}$$

$$\Rightarrow x = \frac{1}{2}\tan^{-1}\frac{2\alpha}{1-\alpha^2-\beta^2}$$

Also $2iy = (x+iy)-(x-iy)$

$$\tan 2iy = \tan[(x+iy)-(x-iy)]$$

$$\Rightarrow \tan 2iy = \frac{\tan(x+iy) - \tan(x-iy)}{1+\tan(x+iy)\cdot\tan(x-iy)}$$

$$= \frac{\alpha + i\beta - \alpha + i\beta}{1+\alpha^2+\beta^2}$$

$$\Rightarrow \tan 2iy = \frac{2i\beta}{1+\alpha^2+\beta^2}$$

$$\Rightarrow \cancel{i}\tanh 2y = \cancel{i}\frac{2\beta}{1+\alpha^2+\beta^2}$$

$$\Rightarrow 2y = \tanh^{-1}\frac{2\beta}{1+\alpha^2+\beta^2}$$

$$\Rightarrow y = \cancel{\frac{1}{2}\tanh^{-1}}\ \frac{1}{2}\tanh^{-1}\frac{2\beta}{1+\alpha^2+\beta^2}$$